

# A transient Markov chain with finitely many cutpoints

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*Dedicated to David Freedman with admiration*

**Abstract:** We give an example of a transient reversible Markov chain that almost surely has only a finite number of cutpoints. We explain how this is relevant to a conjecture of Diaconis and Freedman and a question of Kaimanovich. We also answer Kaimanovich's question when the Markov chain is a nearest-neighbor random walk on a tree.

## 1. Introduction

While studying extensions of De Finetti's theorem to Markov chains, Diaconis and Freedman [3] stated a general conjecture for transient Markov chains  $\{S_n\}$ . We give a result on cutpoints that is relevant to their conjecture. We begin with some background.

We say that an event  $A$  in the space of trajectories of the Markov chain is **exchangeable** if it is invariant under finite permutations, i.e., if  $(S_0, S_1, \dots) \in A$ , then so is  $(S_{\pi(0)}, \dots, S_{\pi(n)}, S_{n+1}, \dots)$  for any  $n$  and any permutation  $\pi$  of  $\{0, \dots, n\}$ . The  $\sigma$ -field of exchangeable events,  $\mathcal{E}$ , is called the exchangeable  $\sigma$ -field. Let  $\bar{\mathcal{E}}$  be the completion of  $\mathcal{E}$ . A transient process visits each state only finitely often, and so for each state  $x$  in the state space  $X$  there is a random variable  $V(x)$  that counts the number of visits,  $V(x) := \#\{n \geq 0; S_n = x\}$ . We call the collection  $V := \{V(x)\}_{x \in X}$  the **occupation numbers** of the process. Clearly,  $V$  is  $\mathcal{E}$ -measurable. A natural question, posed by Kaimanovich [6], is to determine under what conditions the exchangeable  $\sigma$ -field is generated by  $V$ . This was motivated by similar issues arising in the study [7] of random walks on lamplighter groups.

Write  $V_n(x) := \#\{k \in [0, n]; S_k = x\}$ . Note that an event  $A \in \sigma(S_j; j \geq 0)$  is invariant under permutations of  $S_0, \dots, S_n$  if and only if  $A \in \sigma(V_n, S_{n+1}, S_{n+2}, \dots)$ . Therefore

$$(1) \quad \mathcal{E} = \bigcap_n \sigma(V_n, S_{n+1}, S_{n+2}, \dots).$$

\*Supported in part by NSF Grant DMS-04-06017.

†Supported in part by NSF Grant DMS-06-05166.

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AMS 2000 subject classifications: Primary 60J10; secondary 60J50.

Keywords and phrases: birth-and-death chain, cutpoints, exchangeable, nearest-neighbor random walk, occupation numbers, transient Markov chain, trees.

For any Markov chain  $\{S_n\}$ , the sequence of transitions  $\{(S_n, S_{n+1})\}$  is also a Markov chain; for such chains of transitions, Kaimanovich's question was posed earlier as a conjecture by Diaconis and Freedman in [3]. To be precise, let  $M_n(x, y)$  be the number of transitions made from  $x$  to  $y$  up to time  $n$ , so that  $M_n(x, y)$  increases to a finite limit  $M(x, y)$  as  $n \rightarrow \infty$ . They made the following conjecture in [3]:

**Conjecture 1.1.** *The intersection of the  $\sigma$ -fields*

$$(2) \quad \bigcap_n \sigma(M_n, S_{n+1}, S_{n+2}, \dots)$$

*is always generated (up to completion) by  $M$ .*

By comparing (2) to (1), we see that (2) is just the exchangeable  $\sigma$ -field for the chain of transitions  $\{(S_n, S_{n+1})\}$ .

James and Peres [5] related the questions above to *cutpoints* of the Markov chain trajectory. Call  $x$  a **cutpoint** if for some  $k$ , we have  $S_k = x$  and the future of the chain,  $\{S_{k+1}, S_{k+2}, \dots\}$ , is disjoint from its past  $\{S_0, S_1, \dots, S_k\}$ . Call  $S_k$  a **strong cutpoint** if the probability of a transition from  $S_i$  to  $S_j$  is 0 whenever  $i < k < j$ . In [5], Conjecture 1.1 was proved under the condition

(3) the Markov chain  $\{S_n\}$  has infinitely many cutpoints almost surely.

We give a brief outline to illustrate the connection; see [5] for more details. Under the assumption (3), the portions  $\psi_1, \psi_2, \dots$  of the space-time path  $(n, S_n)$  between successive cutpoints are conditionally independent given  $M$ , and the intersection (2) is contained in the tail  $\sigma$ -field of the  $\{\psi_j\}_{j \geq 1}$ , which is trivial (given  $M$ ) by Kolmogorov's zero-one law. Conditional triviality of a  $\sigma$ -field given  $M$  means that the  $\sigma$ -field is generated by  $M$  up to completion.

James and Peres [5] also showed that if  $\{S_n\}$  almost surely has infinitely many strong cutpoints, then  $\mathcal{E}$  is generated by the occupation numbers. Thus, if every transient Markov chain had infinitely many strong cutpoints a.s., then Kaimanovich's question would be resolved.

In general, one expects that a random walk that is “very transient” will have infinitely many strong cutpoints. As shown in [1, 5, 8], transient random walks on Cayley graphs have infinitely many strong cutpoints a.s. More precisely, Lawler [8] proved (3) for simple random walk on the lattices  $\mathbf{Z}^d$  for  $d \geq 4$  and his argument applies to strong cutpoints and to any Cayley graph with volume growth at least polynomial of degree 5. This was extended, using a different argument, to  $\mathbf{Z}^3$  in [5]. Blachère [1] extended the argument of [5] and showed that simple random walks on all transient Cayley graphs of groups have infinitely many strong cutpoints.

This raises the natural question of whether *every* transient Markov chain has infinitely many cutpoints a.s.; a positive answer would establish the conjecture of Diaconis and Freedman. In Section 3 we show, however, that this is not true, even for birth-and-death chains.

## 2. Exchangeability, transition counts and trees

In this section, we show that for transient nearest-neighbor walks on trees, the exchangeable  $\sigma$ -field is generated by the occupation numbers. This result was established in the thesis [4] of the first author, but was never published; the proof

here is shorter than in [4], but relies on the same ideas. Note that the example in Section 3 is a nearest-neighbor random walk on a special tree (a halfline) such that the walk a.s. has finitely many cutpoints, so the proof cannot rely on cutpoints.

Consider a transient Markov chain as in the introduction. If  $V(x) > 0$ , let  $U(x)$  be the state visited by the Markov chain immediately after its last visit to  $x$ . For completeness, define  $U(x) := x$  when  $V(x) = 0$ . Let  $\bar{\sigma}$  denote the completion of a  $\sigma$ -field.

**Theorem 2.1.** *Let  $\{S_n\}$  be a transient Markov chain starting at a fixed state,  $x_0$ . Then  $\mathcal{E} \subseteq \bar{\sigma}(\{M(x, y), U(x); x, y \in X\})$ .*

*Proof.* As in Wilson [9], we imagine running the Markov chain by using infinite stacks under each of the states. The stack under a state  $x$  consists of possible successors to  $x$  and is generated independently of all other stacks by using the transition probabilities from  $x$  repeatedly for independent successors. Once the stacks are generated, the chain moves by moving to the state given at the top of the stack under  $x_0$  and removing (“popping”) the top state under  $x_0$ . This is repeated from the current state, and so on. The number of states under  $x$  that are eventually popped equals  $V(x)$  and the last one is  $U(x)$ . Let  $W(x)$  be the ordered list of states under  $x$  that are popped, *excluding* the last one. Write  $[W(x)]$  for multi-set of states in  $W(x)$ , i.e., the unordered list of states (with repetition) in  $W(x)$ . Note that  $\sigma(M(x, y), U(x); x, y \in X) = \sigma([W(x)], U(x); x \in X)$ .

We first claim that if  $W(x)$  is re-ordered for  $x$  in some finite set of states  $A$ , then the resulting chain  $\{S'_n\}$  starting at  $x_0$  will have the same counts  $M(x, y)$  and same final exits  $U(x)$ . It suffices to prove this when  $A$  is a singleton. Moreover, if  $A$  is not  $x_0$ , then we may simply begin the chain when it first reaches  $A$  and pop the states that are used before then, reducing the situation to  $A = \{x_0\}$ . Thus, let  $A = \{x_0\}$ . The transitions of the chain  $(S_0, S_1, \dots)$  describe an Eulerian circuit of a directed multi-graph,  $G$ . That is,  $G$  consists of directed edges  $(S_k, S_{k+1})$  connecting vertices  $\{S_k\}$  and each vertex has the same number of edges leading to it as leading away from it, except that  $x_0$  has one more edge leading away. When  $W(x_0)$  is re-ordered, the sequence  $(S'_0, S'_1, \dots)$  does not leave  $G$  (while using each edge at most once) since the number of possible arrivals to a vertex via an edge of  $G$  is at most the number of possible departures. Thus,  $(S'_0, S'_1, \dots)$  traverses a subgraph  $G'$  of  $G$ . If we re-order again to the original order, then this argument shows that the resulting graph covered,  $G$ , is a subgraph of  $G'$ . Thus,  $G' = G$ . Therefore, the final transition counts are the same, as claimed. In addition, the stacks were popped in the same order at all vertices other than  $x_0$ , so their final exits are unchanged, as is  $U(x_0)$ .

We next claim that the distribution of  $\{S_n\}$  given  $[W(x)]$  and  $U(x)$  for all  $x \in X$  can be represented as follows: Choose randomly and uniformly an ordering  $W(x)$  for each  $[W(x)]$ , independently for each  $x \in X$ . Then the resulting walk starting from  $x_0$  and determined by these stacks has the same law as the Markov chain. To see this, consider the set  $B$  of trajectories that correspond to a given collection of  $[W(x)]$  and  $U(x)$ . Let  $\{S_n\} \in B$  be one such trajectory. Since re-ordering any finite set of the corresponding  $W(x)$  gives a finite permutation of  $\{S_n\}$  with the same counts and final exits,  $B$  and the conditional Markov chain measure on  $B$  are preserved. Therefore the Markov chain measure is preserved under re-ordering every  $W(x)$ . The only such invariant measure is the one described, so the claim is proved.

Finally, let  $C \in \mathcal{E}$ . Let  $B$  be the set of trajectories that correspond to a given collection of  $[W(x)]$  and  $U(x)$ . Since both  $C$  and  $B$  are invariant under re-ordering any finite  $W(x)$ , so is  $C \cap B$ . In addition, the orderings  $W(x)$  are independent

given all  $[W(x)]$  (and  $U(x)$ ), so the conditional probability of  $C$  given  $B$  is 0 or 1 by Kolmogorov's 0-1 law. Let  $D_0$  be the union of those  $B$  for which the conditional probability of  $C$  given  $B$  is 0 and  $D_1$  be the union of the other  $B$ . Then  $P[C \cap D_0] = 0$ , so  $P[C \triangle D_1] = 0$ . Since  $D_1 \in \bar{\sigma}(M(x, y), U(x); x, y \in X)$ , the theorem is proved.  $\square$

**Corollary 2.1.** *For a transient nearest-neighbor random walk on a tree (with arbitrary transition probabilities), we have  $\bar{\mathcal{E}} = \bar{\sigma}(V)$ .*

*Proof.* Since a transient random walk on a tree  $T$  must tend to some end of  $T$ , it follows that the pointers  $U(x)$  are determined by the occupation field  $V$ . In view of the preceding theorem, it suffices to show that the transition numbers  $M(x, y)$  are also determined by  $V$ . Write  $L_0 = S_0 = x_0$ , and for  $j \geq 1$  define  $L_j = U(L_{j-1})$ . The sequence  $L = \{L_j; j \geq 0\}$  is known as the *loop-erasure* of the trajectory  $\{S_k; k \geq 0\}$ . Consider the finite tree  $T_F = T_F(L_k)$  that is spanned by  $L_k$  and all vertices  $x$  with  $V(x) > 0$  and that can be reached from  $x_0$  without visiting  $L_k$ . The proof will now follow from the following **claim**: *Given a finite walk from  $x_0$  to  $y$  on a finite tree  $T_F$ , the edge transition numbers  $M_F$  of the walk are determined by the occupation numbers  $V_F$  of all vertices except  $y$ .* The claim is proved by induction on the number  $N$  of vertices in  $T_F$ . The base case  $N \leq 2$  is clear. For  $N > 2$ , the tree  $T_F$  has some leaf  $z$  that is different from  $y$ . Let  $z_*$  denote the neighbor of  $z$ . Clearly  $M(z, z_*) = V(z)$  and  $M(z_*, z) = V(z) - \mathbf{1}_{z=x_0}$ . Removing  $z$  from the tree and subtracting  $V_F(z)$  from  $V_F(z_*)$  reduces the problem to a tree with  $N - 1$  vertices and completes the induction step. To apply the claim to our situation, take  $y = L_k$  and observe that for all vertices  $w \in T_F(L_k)$  except possibly  $L_k$  itself, the occupation number  $V(w)$  determined by the infinite random walk path coincides with  $V_F(w)$ , the occupation number determined by the portion of that path in  $T_F(L_k)$ . (It is certainly possible that  $V(L_k) > V_F(L_k)$ , due to excursions of the random walk from  $L_k$  to the complement of  $T_F$ .)  $\square$

### 3. A transient birth-and-death chain with finitely many cutpoints

We shall exhibit a birth-and-death chain, i.e., a nearest-neighbor random walk on  $\mathbb{N}$ , which is transient but has only finitely many cutpoints a.s. We shall use the following basic fact about random walks and electrical networks. Let  $r_k > 0$  be given for  $k \geq 1$ . (Interpret  $r_k$  as the resistance of the edge between  $k$  and  $k + 1$ .) Consider the birth-and-death chain on  $\{1, 2, \dots, n\}$  where the transition probability from 1 to 2 is 1, and for  $k > 1$ , the transition probability from  $k$  to  $k + 1$  is  $r_{k-1}/(r_{k-1} + r_k)$  and the transition probability from  $k$  to  $k - 1$  is  $r_k/(r_{k-1} + r_k)$ . Then the probability that the chain reaches  $n$  before 1 when starting from  $k$  equals  $\sum_{j=1}^{k-1} r_j / \sum_{j=1}^{n-1} r_j$ . See [2], §§II.1 and IX.2. Of course, this can also be phrased as a standard gambler's ruin calculation. In particular, taking a limit as  $n \rightarrow \infty$  shows that transience is equivalent to  $\sum_{j=1}^{\infty} r_j < \infty$ .

**Theorem 3.1.** *Fix  $\beta > 1$ . Let  $r_k > 0$  have the property that  $r_k \asymp k^{-1}(\log k)^{-\beta}$  for all  $k \geq 2$ , where the symbol  $\asymp$  means that the ratio of the two sides is bounded above and below by positive constants that do not depend on  $k$ . Consider the birth-and-death chain on  $\mathbb{N} = \{1, 2, \dots\}$  with transition probability  $r_{k-1}/(r_{k-1} + r_k)$  from  $k$  to  $k + 1$  and transition probability  $r_k/(r_{k-1} + r_k)$  from  $k$  to  $k - 1$  for all  $k \geq 2$ . (The transition probability from 1 to 2 is 1.) Then this chain is transient and has only finitely many cutpoints a.s.*

*Proof.* We may assume the chain starts at 1. Since  $\sum_k r_k < \infty$ , the walk is transient.

Denote  $t_k := \sum_{j \geq k} r_j$ . The usual gambler's ruin calculation shows that the probability that the walk will have  $k$  as a cutpoint is  $p_k = r_k/t_k$ .

Let  $j < k$ . Given that  $k$  is a cutpoint, let  $Q_k(j)$  be the conditional probability that  $j$  is a cutpoint. Then  $Q_k(j)$  is the probability that a walk starting at  $j+1$  visits  $k+1$  before visiting  $j$ , i.e.,

$$(4) \quad Q_k(j) = \frac{r_j}{(t_j - t_{k+1})}.$$

This is also the conditional probability

$$\mathbf{P}[j \text{ is a cutpoint} \mid k \text{ is a cutpoint}, F_{k+1}],$$

where  $F_{k+1}$  is any event determined by the future of the walk after it reaches  $k+1$  for the first time.

Let  $C_{j,k}$  be the set of cutpoints in  $(2^j, 2^k]$  and  $A_{j,k} := |C_{j,k}|$ . Write  $a_m := P[A_{m,m+1} > 0]$  and

$$b_m := \min \left\{ \sum_{i=1}^{2^{m-1}} Q_k(k-i); k \in (2^m, 2^{m+1}] \right\}.$$

On the event that  $A_{m,m+1} > 0$ , let  $\ell_m$  be the largest cutpoint in  $C_{m,m+1}$ . Bound below the expected number of cutpoints in  $(2^{m-1}, 2^{m+1}]$  by conditioning on the last cutpoint in  $(2^m, 2^{m+1}]$ , if there is one:

$$(5) \quad \begin{aligned} \sum_{j=2^{m-1}+1}^{2^{m+1}} p_j &= \mathbf{E}[A_{m-1,m+1}] \\ &\geq a_m \mathbf{E}[A_{m-1,m+1} \mid A_{m,m+1} > 0] \\ &= a_m \mathbf{E}[\mathbf{E}[A_{m-1,m+1} \mid A_{m,m+1} > 0, \ell_m]] \\ &\geq a_m b_m. \end{aligned}$$

Now  $t_j \asymp (\log j)^{-\beta+1}$ , whence  $p_j \asymp (j \log j)^{-1}$  for  $j \geq 2$ . Furthermore, we have  $t_{k-i} - t_{k+1} \asymp i r_k \asymp i r_{k-i}$  for  $1 \leq i \leq 2^{m-1}$  and  $2^m < k \leq 2^{m+1}$ . By (4), this means that  $Q_k(k-i) \geq c/i$  for some constant  $c > 0$  and  $i, k$  in those ranges, which gives in turn that  $b_m \geq c'/m$  for some constant  $c' > 0$ . On the other hand, the left-hand side of (5) is at most  $c''(\log \log 2^{m+1} - \log \log 2^m) \leq c'''/m$  for some  $c'', c''' < \infty$ . It follows that  $a_m = O(1/m^2)$  is summable, so that there are a.s. only finitely many cutpoints by the Borel-Cantelli lemma. It also follows that with positive probability, there are no cutpoints at all.  $\square$

#### 4. Concluding remarks

Given a transient Markov chain  $\{S_j\}$  with a fixed starting state, it is easy to see that for any  $n$ , the event  $A_n$  that  $S_0, S_1, \dots, S_n$  are all cutpoints has positive probability. Indeed, starting from a trajectory  $S_0, S_1, S_2, \dots$ , consider the corresponding loop-erased path  $\{L_j\}$  obtained by erasing cycles in the path as they are created. More precisely,  $L_0 = x_0$  and  $L_j = U(L_{j-1})$  for  $j > 0$ , where  $U(\cdot)$  is the ultimate successor function defined in Section 2. Fix a sequence of vertices  $(x_1, \dots, x_n)$  such that the event  $B_n = \{(L_0, \dots, L_n) = (x_0, \dots, x_n)\}$  has  $P(B_n) > 0$ . If  $B_n$  holds for the

trajectory  $\{S_j^*\}$ , then  $x_j = L_j = S_{k_j}$  for some random sequence  $\{k_j\}$ , and we define a new trajectory  $\{S_j^*\}$  with  $S_j^* = L_j$  for  $j = 0, \dots, n$  and  $S_{n+i}^* = S_{k_n+i}$  for  $i > 0$ . For this new trajectory  $x_0, \dots, x_n$  are all cutpoints. We conclude that  $P(A_n) \geq P(B_n) \prod_{j=1}^n p(x_{j-1}, x_j) > 0$ .

We do not know whether every transient Markov chain has an infinite expected number of cutpoints. For any birth-and-death chain, this does hold since (in the notation of the preceding proof)  $\sum_{k \geq m} p_k \geq \sum_{k \geq m} r_k/t_m = 1$  for every  $m$ , whence the series  $\sum_k p_k$  diverges.

Another natural question that we cannot answer is whether a *simple* random walk on any transient graph of bounded degree must have infinitely many cutpoints a.s.

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